



SCHOOL OF SCIENCE AND TECHNOLOGY
Semester 1, 2026

FINAL EXAMINATION PAPER

MTH315: ABSTRACT ALGEBRA

Important Instructions to Students:

1. Read the questions, instructions carefully and answer the questions.
2. You are allowed an **extra 10 minutes reading time** (during this time, you are not allowed to write with pen) and **180 minutes (3 Hours) to complete this paper**.
3. Silent, non-programmable calculators are allowed, but not supplied.
4. Write all your answers in the answer booklet provided. If you need extra sheets, please ask the supervisor. Write your name, course code and student identification number in the answer booklet and in the attendance sheet.
5. The paper consists of five (5) questions. All questions are **compulsory** to answer.
6. Write all your answers legibly. The marker has the right to award zero for answers that are not legible.
7. This is a **Closed Book Examination**. The University views plagiarism, cheating and dishonest practice as serious offences and if found, offenders are penalized.

Examination Paper Content (Format):

<u>Question</u>	<u>Content</u>	<u>Marks</u>	<u>Suggested Time</u>
1	Groups and Subgroups	20	36 minutes
2	Cyclic Groups	20	36 minutes
3	Permutations, Lagrange Theorem and Product Groups	20	36 minutes
4	Homomorphisms, Kernels and Factor Groups	20	36 minutes
5	Factorisation and Rings	20	36 minutes
	TOTAL	100	180 Minutes

Answer all questions. All questions are compulsory.

Question 1	Start on a new page	[20 marks]
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(a) Define what it means for a non-empty set G with a binary operation $*$ to be a **group**. (2 marks)

(b) Let $S = \{0,1,2,3\}$ and define a binary operation $*$ on S by

$$a * b = (a + b) \bmod 4.$$

Determine whether S is closed under $*$. Justify your answer. (2 marks)

(c) Determine whether the set of even integers under addition forms a group. Give reasons. (2 marks)

(d) Define a subgroup.

(e) Show that $3\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ under addition by verifying the subgroup test. (2 marks)

(f) Let (2 marks)

$$H = \{n \in \mathbb{Z} : n \geq 0\}.$$

Determine whether H is a subgroup of $(\mathbb{Z}, +)$. Give a reason. (2 marks)

(g) Define the binary operation $*$ on $\mathbb{Z}$ by

$$a * b = a + b + 1 \text{ for all } a, b \in \mathbb{Z}.$$

Determine whether $(\mathbb{Z}, *)$ is a group. Justify your answer. (3 marks)

(h) Determine whether $(\mathbb{Z}^+, +)$ forms a group. Hence, state whether it is an abelian group. (3 marks)

(i) What does it mean for two binary structures $(G, *)$ and $(K, \circ)$ to be isomorphic? (2 marks)

Question 2**Start on a new page****[20 marks]**

- (a) Define what it means for a group G to be abelian. (2 marks)
- (b) Define a cyclic group. (2 marks)
- (c) Prove that every cyclic group is abelian. (3 marks)
- (d) List all generators of $\mathbb{Z}_{10}$ and justify your answer. (3 marks)
- (e) Let $\mathbb{Z}_{12}$ be the cyclic group under addition modulo 12. List all generators of $\mathbb{Z}_{12}$. (2 marks)
- (f) Find $\langle 4 \rangle \leq \mathbb{Z}_{12}$ and determine its order. (2 marks)
- (g) Find the subgroup $\langle 6 \rangle$ in $\mathbb{Z}_8$ and determine its order. (2 marks)
- (h) Let $H = \{0, 3, 6, 9\}$ be a subgroup of $\mathbb{Z}_{12}$. Find all distinct cosets of H in $\mathbb{Z}_{12}$. (2 marks)
- (i) Find the index of $\langle 3 \rangle \leq \mathbb{Z}_{15}$ and interpret its meaning in terms of cosets. (2 marks)

Question 3**Start on a new page****[20 marks]**

(a) Let

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 4 & 1 & 3 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 2 & 6 & 5 \end{bmatrix}$$

be three permutations in the symmetric group S_6 .

- (i) Compute $\sigma\mu^{-1}$. (2 marks)
- (ii) Write σ , τ and μ in cycle notation. (3 marks)

- (iii) What are the orbits of τ and μ ? (2 marks)
- (iv) Are τ and μ cycles? Give reasons. (2 marks)
- (v) Write σ and τ as products of transpositions. (2 marks)
- (vi) Classify σ and τ as even or odd permutations. (1 mark)

(b)

- (i) State Lagrange's Theorem. (2 marks)
- (ii) Prove that every group of prime order is cyclic. (3 marks)
- (iii) Find the order of $(4,6)$ in $\mathbb{Z}_8 \times \mathbb{Z}_{10}$. Hence determine whether $(4,6)$ is a generator of the group. (3 marks)

Question 4	Start on a new page	[20 marks]
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- (a) Define a group homomorphism. (2 marks)
- (b) Determine whether $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$, defined by

$$\phi(x) = 4x$$
 is a group homomorphism under addition. (3 marks)
- (c) Let $\phi: \mathbb{Z}_8 \rightarrow \mathbb{Z}_4$, be defined by

$$\phi(x) = x \text{ mod } 4$$
 - (i) Find the image of ϕ (2 marks)
 - (ii) Find $\ker(\phi)$ (2 marks)
 - (iii) Hence determine whether ϕ is injective. (1 mark)

(d) Let

$$H = \langle (0,2) \rangle \leq (\mathbb{Z}_6 \times \mathbb{Z}_4).$$

(i) Find the subgroup H.

(2 marks)

(ii) Determine the order of the factor group

$$(\mathbb{Z}_6 \times \mathbb{Z}_4) / H.$$

(2 marks)

(e) Let $G = \mathbb{Z}_8$ and $H = \langle 2 \rangle$.

(i) Find the subgroup H.

(2 marks)

(ii) List all distinct cosets of H in G.

(2 marks)

(iii) Find $|G/H|$ and interpret its meaning.

(2 marks)

Question 5**Start on a new page****[20 marks]**

(a) Solve the following equation in $\mathbb{Z}_{12}$

$$x^2 - x - 6 = 0.$$

(3 marks)

(b) Explain why $ab=0$ does not necessarily imply $a=0$ or $b=0$ in $\mathbb{Z}_n$. Give one example.

(2 marks)

(c) Evaluate $(x+1)(x+2)$ at $x = 2$ in $\mathbb{Z}_5$

(2 marks)

(d) Let $\phi(f(x)) = f(1)$. Compute

$$\phi(x^2 + 2x + 3)$$

in $\mathbb{Z}_5$.

(2 marks)

(e) Define what it means for a ring R to be an integral domain.

(2 marks)

(f) Prove that every field is an integral domain.

(3 marks)

(g) Factor the polynomial

$$x^2 + 4x + 3$$

in $\mathbb{Z}_5[x]$ and verify your result.

(3 marks)

(h) Determine whether

$$x^2 + 2x + 2$$

is reducible or irreducible in $\mathbb{Z}_5[x]$. Justify your answer.

(3 marks)

THE END